

# ON THETA SERIES ATTACHED TO THE LEECH LATTICE

SHOYU NAGAOKA AND SHO TAKEMORI

ABSTRACT. Some congruence relations satisfied by the theta series associated with the Leech lattice are given.

## 1. INTRODUCTION

The main object of this note is concerned with the degree  $n$  theta series  $\vartheta_A^{(n)}$  associated with a rank  $m$  unimodular lattice  $\Lambda = \Lambda_m$  (or equivalently, degree  $m$  symmetric matrix  $S = S_m$ ). It is known that the theta series  $\vartheta_A^{(n)}$  becomes a Siegel modular form of weight  $\frac{m}{2}$  for  $\Gamma_n := Sp_n(\mathbb{Z})$ . Now we consider the case that  $\Lambda = \mathcal{L}$ : the Leech lattice. The Leech lattice is an even unimodular 24-dimensional lattice (cf. § 2.4). Therefore  $\vartheta_{\mathcal{L}}^{(n)}$  is a Siegel modular form of weight 12 for  $\Gamma_n$ . The main purpose is to show that the theta series  $\vartheta_{\mathcal{L}}^{(2)}$  satisfies the following congruence relation:

$$\Theta(\vartheta_{\mathcal{L}}^{(2)}) \equiv 0 \pmod{23} \quad (\text{Theorem 3.2}).$$

where  $\Theta$  is the theta operator defined in § 2.3. In this note, we present two different kinds of the proof. The first proof depends on the fact that the image of  $\Theta(\vartheta_{\mathcal{L}}^{(2)})$  under the theta operator is congruent to a cusp form (cf. § 3.2). The second proof is based on the fact that  $\Theta(\vartheta_{\mathcal{L}}^{(2)})$  is congruent to the theta series associated with the binary quadratic form of discriminant  $-23$  (cf. § 3.3).

It should be noted that a similar congruence relation appeared in [4]. That is, the following congruence relation was proved:

$$\Theta(X_{35}) \equiv 0 \pmod{23}.$$

where  $X_{35}$  is the Igusa cusp form of weight 35 (cf. § 3.4).

In § 3.5, we introduce another congruence relation satisfied by  $\vartheta_{\mathcal{L}}^{(2)}$ :

$$\vartheta_{\mathcal{L}}^{(2)} \equiv 1 \pmod{13} \quad (\text{Theorem 3.6}).$$

This gives an example of weight  $p-1$  modular form  $F$  satisfying  $F \equiv 1 \pmod{p}$  (e.g. cf. [1]).

## 2. PRELIMINARIES

**2.1. Notation.** First we confirm notation. Let  $\Gamma_n = Sp_n(\mathbb{Z})$  be the Siegel modular group of degree  $n$  and  $\mathbb{H}_n$  the Siegel upper-half space of degree  $n$ . We denote by  $M_k(\Gamma_n)$  the  $\mathbb{C}$ -vector space of all Siegel modular forms of weight  $k$  for  $\Gamma_n$ , and  $S_k(\Gamma_n)$  is the subspace of cusp forms.

Any  $F(Z)$  in  $M_k(\Gamma_n)$  has a Fourier expansion of the form

$$F(Z) = \sum_{0 \leq T \in \text{Sym}_n^*(\mathbb{Z})} a(F; T) q^T, \quad q^T := \exp(2\pi i \text{tr}(TZ)), \quad Z \in \mathbb{H}_n,$$

where

$$\text{Sym}_n^*(\mathbb{Z}) := \{T = (t_{ij}) \in \text{Sym}_n(\mathbb{Q}) \mid t_{ii}, 2t_{ij} \in \mathbb{Z}\}.$$

Namely we write the Fourier coefficient corresponding to  $T \in \text{Sym}_n^*(\mathbb{Z})$  as  $a(F; T)$ .

For a subring  $R$  of  $\mathbb{C}$ , let  $M_k(\Gamma_n)_R \subset M_k(\Gamma_n)$  denote the space of all modular forms whose Fourier coefficients lie in  $R$ .

**2.2. Formal Fourier expansion.** For  $T = (t_{lj}) \in \text{Sym}_n^*(\mathbb{Z})$  and  $Z = (z_{lj}) \in \mathbb{H}_n$ , we write  $q_{lj} := \exp(2\pi i z_{lj})$ . Then

$$q^T = \exp(2\pi i \text{tr}(TZ)) = \prod_{l < j} q_{lj}^{2t_{lj}} \prod_{l=1}^n q_{ll}^{t_{ll}}.$$

Therefore we may consider  $F \in M_k(\Gamma_n)_R$  as an element of the formal power series ring:

$$F = \sum a(F; T) q^T \in R[q_{lj}, q_{lj}^{-1}] \llbracket q_{11}, \dots, q_{nn} \rrbracket.$$

For a prime  $p$ , we denote by  $\mathbb{Z}_{(p)}$  the local ring at  $p$ . For two elements

$$F_i = \sum a(F_i; T) q^T \in \mathbb{Z}_{(p)}[q_{lj}, q_{lj}^{-1}] \llbracket q_{11}, \dots, q_{nn} \rrbracket, \quad (i = 1, 2),$$

we write  $F_1 \equiv F_2 \pmod{p}$  if the congruence

$$a(F_1; T) \equiv a(F_2; T) \pmod{p}$$

satisfies for all  $T \in \text{Sym}_n^*(\mathbb{Z})$ .

**2.3. Theta operator.** For a  $F = \sum a(F; T) q^T \in M_k(\Gamma_n)$ , we associate the formal power series

$$\Theta(F) := \sum a(F; T) \cdot \det(T) q^T \in \mathbb{C}[q_{lj}, q_{lj}^{-1}] \llbracket q_{11}, \dots, q_{nn} \rrbracket.$$

It should be noted that  $\Theta(F)$  is not necessarily modular form. However the following fact holds.

**Theorem 2.1.** (Böcherer-Nagaoka [1], Theorem 4). Assume that a prime  $p$  satisfies  $p \geq n + 3$ . Then, for any modular form  $F \in M_k(\Gamma_n)_{\mathbb{Z}_{(p)}}$ , there exists a Siegel cusp form  $G \in S_{k+p+1}(\Gamma_n)_{\mathbb{Z}_{(p)}}$  satisfying

$$\Theta(F) \equiv G \pmod{p}$$

as formal power series.

In the case  $n = 1$ , this operator was studied by Ramanujan and played an important role in the theory of  $p$ -adic elliptic modular forms ([8]).

**2.4. Theta series and Leech lattice.** As usual, for a positive matrix  $S \in 2\text{Sym}_m^*(\mathbb{Z})$ , we associate the theta series on  $\mathbb{H}_n$ :

$$\vartheta_S^{(n)}(Z) := \sum_{X \in M_{m,n}(\mathbb{Z})} \exp(\pi i \text{tr}(S[X]Z))$$

where  $S[X] = {}^t X S X$ . It is well-known that

$$\vartheta_S^{(n)} \in M_{\frac{m}{2}}(\Gamma_n)_{\mathbb{Z}}$$

if  $S$  is unimodular.

Let  $\mathcal{L}$  be the Leech lattice (we identify it with the Gram matrix). The Leech lattice is the unique lattice of rank 24, which contains no roots (e.g. cf. [5], Theorem 4.1). From this fact, for example, we see that

$$a\left(\vartheta_{\mathcal{L}}^{(2)}; \begin{pmatrix} m & \frac{r}{2} \\ \frac{r}{2} & 1 \end{pmatrix}\right) = a\left(\vartheta_{\mathcal{L}}^{(2)}; \begin{pmatrix} 1 & \frac{r}{2} \\ \frac{r}{2} & n \end{pmatrix}\right) = 0.$$

**Example 2.2.** It is known that

$$\begin{aligned} \vartheta_{\mathcal{L}}^{(1)} &= (E_4^{(1)})^3 - 720\Delta \\ &= 1 + 196560q^2 + 16773120q^3 + 398034000q^4 + \dots \in M_{12}(\Gamma_1)_{\mathbb{Z}}, \\ &\text{(e.g. cf. [7])}, \end{aligned}$$

where  $\Delta$  is the cusp form of weight 12 defined by

$$\begin{aligned} \Delta &:= \frac{1}{1728}((E_4^{(1)})^3 - (E_6^{(1)})^2) \\ &= q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - 6048q^6 - \dots \in S_{12}(\Gamma_1)_{\mathbb{Z}}. \end{aligned}$$

**2.5. Sturm-type Theorem.** A Sturm-type theorem maintains that, if some of Fourier coefficients of  $F$  vanish mod  $p$ , then  $F$  is congruent zero mod  $p$ .

In the following, for simplicity, we use the abbreviation

$$[m, r, n] := \begin{pmatrix} m & \frac{r}{2} \\ \frac{r}{2} & n \end{pmatrix}, \quad (m, n, r \in \mathbb{Z}).$$

**Theorem 2.3.** (D. Choi, Y. Choie, T. Kikuta [3]). Let  $p \geq 5$  be a prime. Suppose that  $F(Z) \in M_k(\Gamma_2)_{\mathbb{Z}_{(p)}}$  has the Fourier expansion

$$F(Z) = \sum_{0 \leq T=[m,r,n]} a([m, r, n]) q^T.$$

If

$$a([m, r, n]) \equiv 0 \pmod{p}$$

for any  $m, n$  such that

$$0 \leq m \leq \frac{k}{10} \quad \text{and} \quad 0 \leq n \leq \frac{k}{10},$$

then  $F \equiv 0 \pmod{p}$ .

**Remark 2.4.** In [3], the result was proved under more general situation.

**Corollary 2.5.** Suppose that  $F \in M_{12}(\Gamma_2)_{\mathbb{Z}}$  satisfies

$$a(F; T) = 0$$

for any  $0 \leq T \in \text{Sym}_2^*(\mathbb{Z})$  with  $\text{tr}(T) \leq 2$ , then  $F = 0$ .

*Proof.* We can apply Theorem 2.3 to the case  $k = 12$  and infinitely many  $p$ . □

**Corollary 2.6.** Suppose that  $F \in M_{36}(\Gamma_2)_{\mathbb{Z}}$  satisfies

$$a(F; T) \equiv 0 \pmod{23}$$

for any  $0 \leq T \in \text{Sym}_2^*(\mathbb{Z})$  with  $\text{tr}(T) \leq 6$ , then  $F \equiv 0 \pmod{23}$ .

**2.6. Congruences for binary theta series.** Assume that  $p$  is a prime with  $p \equiv 3 \pmod{4}$ . Then there exists  $S \in \text{Sym}_2^*(\mathbb{Z})$  such that

$$\det(2S) = p,$$

namely, the discriminant of  $S$  is  $-p$ . For this  $S$ , we associate the theta series  $\vartheta_S^{(2)}$ . Then  $\vartheta_S^{(2)}$  is a weight 1 modular form for

$$\Gamma_0^2(p) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_2 \mid C \equiv 0_2 \pmod{p} \right\}$$

with character  $\chi_p = \left(\frac{-p}{*}\right)$ . Namely

$$\vartheta_S^{(2)} \in M_1(\Gamma_0^2(p), \chi_p).$$

The following statement is a special case of a result of Böcherer and Nagaoka (cf. [2], Theorem 5).

**Theorem 2.7.** (S. Böcherer, S. Nagaoka) Assume that  $p \geq 7$  and  $p \equiv 3 \pmod{4}$ . Let  $S \in \text{Sym}_2^*(\mathbb{Z})$  be a positive definite binary quadratic form with  $\det(2S) = p$  (i.e. discriminant of  $S = -p$ .) Then there exists a modular form  $G \in M_{\frac{p+1}{2}}(\Gamma_2)_{\mathbb{Z}_{(p)}}$  such that

$$\vartheta_S^{(2)} \equiv G \pmod{p}.$$

## 3. MAIN RESULT

**3.1. Statement of the main result.** As we stated in Introduction, the main purpose of this note is to show that the theta series associated with the Leech lattice satisfies a congruence relation.

**Theorem 3.1.** Let  $a(\vartheta_{\mathcal{L}}^{(2)}; T)$  denote the Fourier coefficient of  $\vartheta_{\mathcal{L}}^{(2)}$ . If  $\det(T) \not\equiv 0 \pmod{23}$ , then

$$a(\vartheta_{\mathcal{L}}^{(2)}; T) \equiv 0 \pmod{23},$$

or equivalently,

$$\Theta(\vartheta_{\mathcal{L}}^{(2)}) \equiv 0 \pmod{23}.$$

**3.2. The first proof.** In this subsection we present a proof of Theorem 3.1 using a property of the theta operator.

*Proof.* We apply Theorem 2.1 in the case that

$$F = \vartheta_{\mathcal{L}}^{(2)} \quad \text{and} \quad p = 23.$$

From this, we can find a Siegel cusp form  $G \in S_{36}(\Gamma_2)$  such that

$$\Theta(\vartheta_{\mathcal{L}}^{(2)}) \equiv G \pmod{23}.$$

By the Table 4.2 in §4, we can confirm

$$a(\Theta(\vartheta_{\mathcal{L}}^{(2)}); T) \equiv a(G; T) \equiv 0 \pmod{23}$$

for any  $0 \leq T \in \text{Sym}_2^*(\mathbb{Z})$  with  $\text{tr}(T) \leq 6$ . Then, by Corollary 2.6, we obtain

$$G \equiv 0 \pmod{23}.$$

This means that

$$\Theta(\vartheta_{\mathcal{L}}^{(2)}) \equiv 0 \pmod{23}.$$

□

**3.3. The second proof.** In this subsection, we give the second proof of our main theorem, which is based on a congruence between theta series.

**Theorem 3.2.** The following congruence relation holds.

$$\vartheta_{[2,1,3]}^{(2)} \equiv \vartheta_{\mathcal{L}}^{(2)} \pmod{23},$$

or equivalently,

$$\Theta(\vartheta_{\mathcal{L}}^{(2)}) \equiv 0 \pmod{23}.$$

*Proof.* We apply Theorem 2.7 in the case  $p = 23$ . Then we see that there is a modular form  $G \in M_{12}(\Gamma_2)_{\mathbb{Z}_{(23)}}$  such that

$$\vartheta_{[2,1,3]}^{(2)} \equiv G \pmod{23}.$$

By the Tables 4.2 in §4, we can confirm that

$$a(\vartheta_{[2,1,3]}^{(2)}; T) \equiv a(G; T) \equiv a(\vartheta_{\mathcal{L}}^{(2)}; T) \pmod{23}$$

for any  $0 \leq T \in \text{Sym}_2^*(\mathbb{Z})$  with  $\text{tr}(T) \leq 6$ . This shows that

$$G \equiv \vartheta_{\mathcal{L}}^{(2)} \pmod{23}.$$

Since

$$\Theta(\vartheta_{[2,1,3]}^{(2)}) \equiv 0 \pmod{23},$$

we obtain

$$\Theta(\vartheta_{\mathcal{L}}^{(2)}) \equiv 0 \pmod{23}.$$

□

**Remark 3.3.** In the degree one case, we have already known the congruence

$$\vartheta_{[2,1,3]}^{(1)} \equiv \vartheta_{\mathcal{L}}^{(1)} \pmod{23},$$

([7], p.3).

**3.4. Igusa's generators.** The theta series  $\vartheta_{\mathcal{L}}^{(2)}$  is a weight 12 Siegel modular form with integral Fourier coefficients. Therefore it can be expressed as a polynomial with Igusa's generators of the ring of Siegel modular forms of degree two over  $\mathbb{Z}$ . In this subsection, we give the explicit form.

Let

$$M(\Gamma_2)_{\mathbb{Z}} = \bigoplus_{k \in \mathbb{Z}} M_k(\Gamma_2)_{\mathbb{Z}}.$$

be the graded ring of Siegel modular forms of degree 2 over  $\mathbb{Z}$ . Igusa [6] constructed a minimal set of generators of this ring. The set of generators consists of fifteen modular forms

$$X_4, X_6, X_{10}, Y_{12}, X_{12}, \dots, X_{48}$$

where subscripts denote their weights. Here the first two modular forms  $X_k$  ( $k = 4, 6$ ) are the weight  $k$  Siegel-Eisenstein series:

$$X_4 = E_4^{(2)}, \quad X_6 = E_6^{(2)}.$$

The modular form  $X_{35}$  appearing in Introduction is one of these generators, moreover, it is the unique odd weight generator.

**Example 3.4.** We give the Fourier expansions of the first five generators:

$$\begin{aligned} X_4 &= 1 + 240(q_{11} + q_{22}) + 2160(q_{11}^2 + q_{22}^2) + (30240 + 13440 \cdot c_1 + 240 \cdot c_2)q_{11}q_{22} \\ &\quad + 6720(q_{11}^3 + q_{22}^3) + (181440 + 138240 \cdot c_1 + 30240 \cdot c_2)(q_{11}^2q_{22} + q_{11}q_{22}^2) + \dots \\ X_6 &= 1 - 504(q_{11} + q_{22}) - 16632(q_{11}^2 + q_{22}^2) + (166320 + 44352 \cdot c_1 - 504 \cdot c_2)q_{11}q_{22} \\ &\quad - 122976(q_{11}^3 + q_{22}^3) + (3792096 + 2128896 \cdot c_1 + 166320 \cdot c_2)(q_{11}^2q_{22} + q_{11}q_{22}^2) + \dots \\ X_{10} &= (-2 + c_1)q_{11}q_{22} + (36 - 16 \cdot c_1 - 2 \cdot c_2)(q_{11}^2q_{22} + q_{11}q_{22}^2) \\ &\quad + (-272 + 99 \cdot c_1 + 36 \cdot c_2 + c_3)(q_{11}^3q_{22} + q_{11}q_{22}^3) + \dots \\ X_{12} &= (10 + c_1)q_{11}q_{22} + (-132 - 88 \cdot c_1 + 10 \cdot c_2)(q_{11}^2q_{22} + q_{11}q_{22}^2) \\ &\quad + (736 + 1275 \cdot c_1 - 132 \cdot c_2 + c_3)(q_{11}^3q_{22} + q_{11}q_{22}^3) + \dots \\ Y_{12} &= (q_{11} + q_{22}) - 24(q_{11}^2 + q_{22}^2) + (1206 + 116 \cdot c_1 + c_2)q_{11}q_{22} \\ &\quad + 252(q_{11}^3 + q_{22}^3) + (115236 + 22176 \cdot c_1 + 1206 \cdot c_2)(q_{11}^2q_{11} + q_{11}q_{22}^2) + \dots, \end{aligned}$$

where  $c_i = q_{12}^i + q_{12}^{-i}$ . (We have more extended expression for each modular form.)

Here we should remark that

$$\Phi(X_4) = E_4^{(1)}, \quad \Phi(X_6) = E_6^{(1)}, \quad \Phi(X_{10}) = \Phi(X_{12}) = 0, \quad \Phi(Y_{12}) = \Delta,$$

where  $\Phi$  is the Siegel operator.

**Theorem 3.5.** Let  $\vartheta_{\mathcal{L}}^{(2)}$  is the degree 2 theta series associated with the Leech lattice  $\mathcal{L}$ . The we have

$$\vartheta_{\mathcal{L}}^{(2)} = X_4^3 - 720Y_{12} + 43200X_{12}.$$

*Proof.* As a matter of course, Theorem 3.5 can be obtained by the direct calculations of the Fourier coefficients of  $X_4$ ,  $X_6$ ,  $X_{12}$ , and  $Y_{12}$ . By Igusa's structure theorem over  $\mathbb{Z}$ , we can write as

$$\vartheta_{\mathcal{L}}^{(2)} = a_1X_4^3 + a_2X_6^2 + a_3X_{12} + a_4Y_{12},$$

with  $a_i \in \mathbb{Z}$  ( $1 \leq i \leq 4$ ). By comparing the Fourier coefficients of both sides (cf. Example 3.4 and §4), we obtain

$$a_1 = 1, \quad a_2 = 0, \quad a_3 = 43200, \quad a_4 = -720.$$

□

**3.5. Another congruence.** In this subsection, we introduce another congruence satisfied by the theta series  $\vartheta_{\mathcal{L}}^{(2)}$ .

**Theorem 3.6.** The following congruence relation holds.

$$\vartheta_{\mathcal{L}}^{(2)} \equiv 1 \pmod{13}.$$

*Proof.* It is known that the weight 12 Siegel Eisenstein series  $E_{12}^{(2)}$  has the property

$$E_{12}^{(2)} \equiv 1 \pmod{13}.$$

On the other hand, we can confirm that

$$a(\vartheta_{\mathcal{L}}^{(2)}; o_2) = 1 \quad \text{and} \quad a(\vartheta_{\mathcal{L}}^{(2)}; T) \equiv 0 \pmod{13}$$

for  $0 \leq T \in \Lambda_2$  with  $\text{tr}(T) \leq 6$ . This shows that

$$\vartheta_{\mathcal{L}}^{(2)} \equiv E_{12}^{(2)} \equiv 1 \pmod{13}.$$

□

**Remark 3.7.** Of course, the congruence in Theorem 3.6, means that

$$a(\vartheta_{\mathcal{L}}^{(2)}; T) \equiv 0 \pmod{13}$$

for any  $0_2 \neq T \in \Lambda_2$ .

#### 4. NUMERICAL EXAMPLES

In this section, we present numerical examples of the Fourier coefficients of  $\vartheta_{\mathcal{L}}^{(2)}$  and  $\vartheta_{[2,1,3]}^{(2)}$ , which is used in our proof of the main results.

**Example 4.1.** Fourier expansion of  $\vartheta_{\mathcal{L}}^{(2)}$

$$\begin{aligned} \vartheta_{\mathcal{L}}^{(2)} = & 1 + 196560(q_{11}^2 + q_{22}^2) \\ & + 16773120(q_{11}^3 + q_{22}^3) \\ & + 398034000(q_{11}^4 + q_{22}^4) \\ & + (18309564000 + 9258762240 \cdot c_1 + 904176000 \cdot c_2 + 196560 \cdot c_4)q_{11}^2 q_{22}^2 \\ & + 4629381120(q_{11}^5 + q_{22}^5) \\ & + (1273079808000 + 815173632000 \cdot c_1 + 187489935360 \cdot c_2 + 9258762240 \cdot c_3) \\ & \cdot (q_{11}^3 q_{22}^2 + q_{11}^2 q_{22}^3) \\ & + 34417656000(q_{11}^6 + q_{22}^6) \\ & + (26182676520000 + 18748993536000 \cdot c_1 + 6444966528000 \cdot c_2 + 815173632000 \cdot c_3 + 18309564000 \cdot c_4) \\ & \cdot (q_{11}^4 q_{22}^2 + q_{11}^2 q_{22}^4) \\ & + (88768382976000 + 65996457246720 \cdot c_1 + 25779866112000 \cdot c_2 + 4320755712000 \cdot c_3 + 187489935360 \cdot c_4 \\ & + 16773120 \cdot c_6)q_{11}^3 q_{22}^3 + \cdots, \end{aligned}$$

where  $c_i := q_{12}^i + q_{12}^{-i}$ .

**Table 4.2.** Fourier coefficients of  $\vartheta_{\mathcal{L}}^{(2)}$  and  $\vartheta_{[2,1,3]}^{(2)}$

| $T = [m, r, n]$ | $\text{tr}(T)$ | $a(\vartheta_{[2,1,3]}^{(2)}; T)$ | $a(\vartheta_{\mathcal{L}}^{(2)}; T)$  |
|-----------------|----------------|-----------------------------------|--|
| [0,0,0]         | 0              | 1                                 | 1  |
| [1,0,0]         | 1              | 0                                 | 0  |
| [2,0,0]         | 2              | 2                                 | $196560 = 2^4 \cdot 3^3 \cdot 5 \cdot 7 \cdot 13$  |
| [1,1,1]         | 2              | 0                                 | 0  |
| [1,0,1]         | 2              | 0                                 | 0  |
| [3,0,0]         | 3              | 2                                 | $16773120 = 2^{12} \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$   |
| [2,2,1]         | 3              | 0                                 | 0  |
| [2,1,1]         | 3              | 0                                 | 0  |
| [2,0,1]         | 3              | 0                                 | 0  |
| [4,0,0]         | 4              | 2                                 | $398034000 = 2^4 \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 13$   |
| [3,3,1]         | 4              | 0                                 | 0  |
| [3,2,1]         | 4              | 0                                 | 0  |
| [3,1,1]         | 4              | 0                                 | 0  |
| [3,0,1]         | 4              | 0                                 | 0  |
| [2,4,2]         | 4              | 2                                 | $196560 = 2^4 \cdot 3^3 \cdot 5 \cdot 7 \cdot 13$  |
| [2,3,2]         | 4              | 0                                 | 0  |
| [2,2,2]         | 4              | 0                                 | $904176000 = 2^7 \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot \underline{23}$                    |
| [2,1,2]         | 4              | 0                                 | $9258762240 = 2^{15} \cdot 3^3 \cdot 5 \cdot 7 \cdot 13 \cdot \underline{23}$                  |
| [2,0,2]         | 4              | 0                                 | $18309564000 = 2^5 \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 13 \cdot \underline{23}$                  |
| [5,0,0]         | 5              | 0                                 | $4629381120 = 2^{14} \cdot 3^3 \cdot 5 \cdot 7 \cdot 13 \cdot \underline{23}$                  |
| [4,4,1]         | 5              | 0                                 | 0  |
| [4,3,1]         | 5              | 0                                 | 0  |
| [4,2,1]         | 5              | 0                                 | 0  |
| [4,1,1]         | 5              | 0                                 | 0  |
| [4,0,1]         | 5              | 0                                 | 0  |
| [3,4,2]         | 5              | 0                                 | 0  |
| [3,3,2]         | 5              | 0                                 | $9258762240 = 2^{15} \cdot 3^3 \cdot 5 \cdot 7 \cdot 13 \cdot \underline{23}$                  |
| [3,2,2]         | 5              | 0                                 | $187489935360 = 2^{13} \cdot 3^7 \cdot 5 \cdot 7 \cdot 13 \cdot \underline{23}$                |
| [3,1,2]         | 5              | 2                                 | $815173632000 = 2^{15} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 13$                                   |
| [3,0,2]         | 5              | 0                                 | $1273079808000 = 2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13 \cdot \underline{23}$    |
| [6,0,0]         | 6              | 2                                 | $34417656000 = 2^6 \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 17 \cdot 103$                    |
| [5,4,1]         | 6              | 0                                 | 0  |
| [5,3,1]         | 6              | 0                                 | 0  |
| [5,2,1]         | 6              | 0                                 | 0  |
| [5,1,1]         | 6              | 0                                 | 0  |
| [5,0,1]         | 6              | 0                                 | 0  |
| [4,5,2]         | 6              | 0                                 | 0  |
| [4,4,2]         | 6              | 0                                 | $18309564000 = 2^5 \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 13 \cdot \underline{23}$                  |
| [4,3,2]         | 6              | 2                                 | $815173632000 = 2^{15} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 13$                                   |
| [4,2,2]         | 6              | 0                                 | $6444966528000 = 2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13 \cdot \underline{23}$    |
| [4,1,2]         | 6              | 0                                 | $18748993536000 = 2^{15} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 13 \cdot \underline{23}$            |
| [4,0,2]         | 6              | 0                                 | $26182676520000 = 2^6 \cdot 3^7 \cdot 5^4 \cdot 7 \cdot 11 \cdot 13^2 \cdot \underline{23}$    |
| [3,6,3]         | 6              | 2                                 | $16773120 = 2^{12} \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$   |
| [3,5,3]         | 6              | 0                                 | 0  |
| [3,4,3]         | 6              | 0                                 | $187489935360 = 2^{13} \cdot 3^7 \cdot 5 \cdot 7 \cdot 13 \cdot \underline{23}$                |
| [3,3,3]         | 6              | 0                                 | $4320755712000 = 2^{18} \cdot 3^2 \cdot 5^3 \cdot 7^2 \cdot 13 \cdot \underline{23}$           |
| [3,2,3]         | 6              | 0                                 | $25779866112000 = 2^{12} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13 \cdot \underline{23}$   |
| [3,1,3]         | 6              | 0                                 | $65996457246720 = 2^{18} \cdot 3^7 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot \underline{23}$     |
| [3,0,3]         | 6              | 0                                 | $88768382976000 = 2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13^2 \cdot \underline{23} \cdot 59$ |

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Shoyu Nagaoka

Department of Mathematics

Kinki University

Higashi-Osaka, Osaka 577-8502, Japan

Email:nagaoka@math.kindai.ac.jp

and

Sho Takemori

Department of Mathematics

Hokkaido University

Kita 10, Nishi 8, Kita-Ku

Sapporo, Hokkaido, 060-0810, Japan

E-mail: stakemorii@gmail.com

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